Qualifying Quiz

Following Qualifying Quiz is a set of 7 demanding problems. Try to solve all of them and submit your solutions in PDF via our application system at mathsbeyondlimits.eu/mysolutions. The deadline for submitting solutions is **April 30, 2017**. Don't get upset if you find them difficult as they are meant to be demanding, thought-provoking and getting the best out of you. Also don't hesitate to submit just partial solutions as sometimes they may be very near the completion. At the same time we discourage you from googling solutions to these problems.

Problem **1**. Determine all non-zero integers a, for which there exists an integer b such that: $ab|a^2+b^2-a$.

Problem **2**. Consider a positive integer n and the function $f:\mathbb{N}\to\mathbb{N}$ by

$$f(x)=\begin{cases}\frac{x}{2} & \text{if } x \text{ is even}\\ \frac{x-1}{2}+2^{n-1} & \text{if } x \text{ is odd}\end{cases}$$

Determine the set

$$A=\{x\in\mathbb{N}|(\underbrace{f\circ f\circ\ldots\circ f}_{n})(x)=x\}.$$

Problem **3**. Let ω be the incircle of $\triangle ABC$. Denote by I the center of ω. The circle with radius AI and center A intersects circumcircle of $\triangle ABC$ at points P and Q. Prove that line PQ is tangent to ω.

Problem **4**. There are 130212 cities in the country of MBL, some pairs of cities are connected by train tracks. Furthermore, you can go between any two cities only by (perhaps multiple) train connections. Prove that the government of MBL may order modernisation of some train tracks so that every city will have an odd number of modern train tracks ending in it.

Problem **5**. Let $\mathbb{N}$ denote the set of positive integers. Find all functions $f:\mathbb{N}\to\mathbb{N}$ such that:

(i) The greatest common divisor of the sequence $f(1), f(2), \ldots$ is 1.

(ii) For all sufficiently large integers[1] n, we have $f(n)\neq 1$ and

$$f(a)^n \mid f(a+b)^{a^{n-1}}-f(b)^{a^{n-1}}$$

for all positive integers a and b.

Problem **6**. Denote by ω the incircle of $\triangle ABC$. Suppose ω touches the sides BC, AC, AB at the points D, E, F respectively. Let I be the center of ω, the segment BI intersects ω at point M. The line tangent to ω at the point M intersects the line AC at the point R. Let the incircle of $\triangle DEF$ touches the segments EF, FD at points D', E' respectively. The line going through A and perpendicular to ID' intersects line going through B and perpendicular to IE' at the point T. Let J be the A-excenter of $\triangle ABC$. Prove that the points R, T, J are collinear.

Problem **7**. Let $x_1, x_2, \ldots, x_n$ be real numbers satisfying $x_1^2+x_2^2+\ldots+x_n^2=1$. Prove that for every integer $k\geq 2$ there are integers $a_1, a_2, \ldots, a_n$, not all zero, such that $|a_i|\leq k-1$ for all i, and $|a_1x_1+a_2x_2+\ldots+a_nx_n|\leq\frac{(k-1)\sqrt{n}}{k^n-1}$.

[1] that is, there exists positive integer N such that for all positive integers $n>N$ this holds.

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